UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
FACULTAD DE INGENIERÍA
DIVISIÓN DE CIENCIAS BÁSICAS
COORDINACIÓN DE MATEMÁTICAS

CÁLCULO INTEGRAL
SEGUNDO EXAMEN FINAL COLEGIADO

TIPO " A "

6 de junio de 2011 ***Semestre 2011-2***

INSTRUCCIONES: Leer cuidadosamente los enunciados de los **7 reactivos** que componen el examen antes de empezar a resolverlos. La duración máxima del examen es de **2.5 horas**.

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1. Calcular el valor de b tal que $$\int_0^b \frac{dx}{\sqrt{1-x}} = 1$$

10 Puntos

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2. Determinar si la integral converge o diverge

$$\int_0^1 (x-1)^{-\frac{1}{3}}\, dx$$

10 Puntos

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3. Efectuar:

a) $\int x \cos 2x \, dx$ b) $\int \frac{\sqrt{1-x^2}}{x^2} dx$ c) $\int \frac{x-3}{x^3+x^2}\, dx$

24 Puntos

4. Calcular el volumen del sólido de revolución que se obtiene al hacer girar alrededor del eje X, la región limitada por las gráficas de ecuación $y = x^2 + 1$ y $y = x + 3$

10 Puntos

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5. Por medio de integrales calcular la longitud de una circunferencia de radio **r**.

10 Puntos

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6. Sea $z = 4e^x \, ln \, y$, $x = ln\left(u \; cos \, v\right)$, $y = u \; sen\left(v\right)$ determine

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u = 2 \\ v = \frac{\pi}{4}}}$$

16 Puntos

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7. En cierta región del espacio, el potencial eléctrico V está dado por la ecuación

$$V\left(x, y, z\right) = 5x^2 - 3\,x\,y + x\,y\,z$$

a) Determine la razón de cambio del potencial V en el punto $P\left(3, 4, 5\right)$, en la dirección del vector $a = i + j - k$.

b) ¿En qué dirección cambia V más rápidamente en el punto P?

20 Puntos

5. Sea

$$\int_0^b \frac{dx}{\sqrt{1-x}} = 1$$

$$-2\sqrt{1-x}\ \Big]_0^b = 1 \qquad \Rightarrow \qquad -2\left[\sqrt{1-b} - \sqrt{1-0}\ \right] = 1$$

$$\sqrt{1-b} - 1 = -\frac{1}{2} \qquad \Rightarrow \qquad \sqrt{1-b} = \frac{1}{2}$$

$$1-b = \frac{1}{4} \qquad \Rightarrow \qquad \boxed{b = \frac{3}{4}}$$

10 Puntos

6.

$$\int_0^1 (x-1)^{-\frac{1}{3}}\ dx$$

$$\int_0^1 (x-1)^{-\frac{1}{3}}\ dx = \lim_{w\to 1^-} \int_0^w (x-1)^{-\frac{1}{3}}\ dx$$

$$\lim_{w\to 1^-}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^w = \frac{3}{2}\lim_{x\to 1^-}\left[\sqrt[3]{(w-1)^2}-\sqrt[3]{(0-1)^2}\right]$$

$$\boxed{\int_0^1 (x-1)^{-\frac{1}{3}}\,dx = -\frac{3}{2}} \qquad \therefore \quad La \;\; \text{integral} \;\; converge$$

10 Puntos

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7. a)

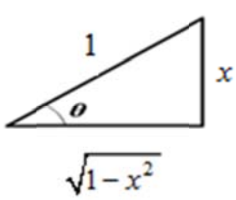
$$I = \int x\cos 2x\,dx = \frac{1}{2}x\,sen\,2x - \frac{1}{2}\int sen\,2x\,dx$$

$$u = x \qquad\qquad dv = \cos 2x$$

$$du = dx \qquad\qquad v = \frac{1}{2}sen\,2x$$

$$\boxed{I = \frac{1}{2}x\,sen\,2x + \frac{1}{4}\cos 2x + C}$$

b) $I = \displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$



$$sen\theta = x \qquad\qquad dx = \cos\theta\,d\theta$$

$$\cos\theta = \sqrt{1-x^2}$$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$I=\int \frac{\cos^2\theta}{sen^2\theta}\,d\theta=\int \csc^2\theta\,d\theta-\int d\theta$$

$$I=-\cot\theta-\theta+C$$

$$I=-\frac{\sqrt{1-x^2}}{x}-angsen(x)+C$$

$$\boxed{\int \frac{\sqrt{1-x^2}}{x}dx=-\frac{\sqrt{1-x^2}}{x}-angsen(x)+C}$$

c) Por fracciones parciales

$$I=\int \frac{x-3}{x^2(x+1)}dx=4\int \frac{dx}{x}-3\int x^{-2}dx-4\int \frac{dx}{x+1}$$

$$\frac{x-3}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$x-3=Ax(x+1)+B(x+1)+Cx^2$$

$$x-3=(A+C)x^2+(A+B)x+B$$

$A+C=0$ $\qquad \boxed{C=-4}$

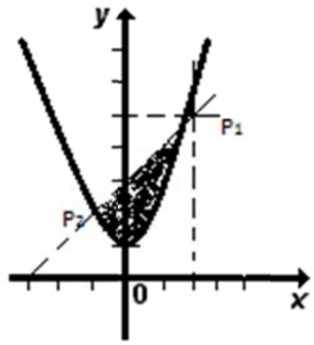
$A+B=1$ $\quad ; \quad \boxed{A=4}$

$\boxed{B=-3}$

$$\boxed{I=4ln|x|+\frac{3}{x}-4ln(x+1)+\ C}$$

24 Puntos

8. Puntos de intersección



$$x^2+1=x+3$$

$$x^2-x-2=0$$

$$x_1=2 \quad ; \; y_1=5$$

$$x_2=-1\, ; \; y_2=2$$

$$P_1(2,5),\; P_2(-1,2)$$

$$V=\pi\int_a^b\left\{\left[R(x)\right]^2-\left[r(x)\right]^2\right\}dx=\pi\int_{-1}^{2}\left[(x+3)^2-\left(x^2+1\right)^2\right]dx$$

$$V=\pi\int_{-1}^{2}\left(-x^4-x^2+6x+8\right)dx=\pi\left[-\frac{x^5}{5}-\frac{x^3}{3}+3x^2+8x\right]_{-1}^{2}$$

$$\boxed{V=\frac{117}{5}\pi u^3}$$

10 Puntos

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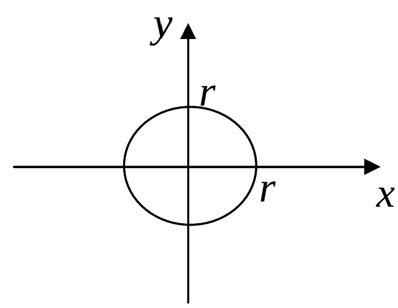
8. Sea



$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$f'(x)=-\frac{x}{\sqrt{r^2-x^2}}$$

$$L=4\int_0^r\sqrt{1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,dx=4\int_0^r\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$L=4\int_0^r\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$L = 4r\, angsen\left(\frac{x}{r}\right)\Bigg|_0^r = 4r\left[angsen(1) - angsen(0)\right]$$

$$L = 4r\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{L = 2\pi r\ [unidades]}$$

10 Puntos

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9. Sea

$$z = 4e^x \ln y, \qquad x = \ln(u\cos v), \qquad y = u\, sen(v)$$

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u=2\\ v=\frac{\pi}{4}}} \qquad por \quad lo \quad que \qquad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$x = \ln\left(2\frac{\sqrt{2}}{2}\right) = \ln\left(\sqrt{2}\right) \qquad y = 2\frac{\sqrt{2}}{2} = \sqrt{2}$$

$$\frac{\partial z}{\partial x} = 4e^x \ln y \quad \Rightarrow \quad \left.\frac{\partial z}{\partial x}\right|_{\substack{u=2\\ v=\frac{\pi}{4}}} = 4e^{\ln\left(\sqrt{2}\right)} \ln\left(\sqrt{2}\right) = 4\sqrt{2}\ln\left(\sqrt{2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{4e^x}{y} \quad \Rightarrow \quad \left.\frac{\partial z}{\partial y}\right|_{\substack{u=2\\ v=\frac{\pi}{4}}} = \frac{4e^{\ln\left(\sqrt{2}\right)}}{\sqrt{2}} = \frac{4\sqrt{2}}{\sqrt{2}} = 4$$

$$\frac{\partial x}{\partial u} = \frac{\cos v}{u\cos v} = \frac{1}{u} \Rightarrow \quad \left.\frac{\partial x}{\partial u}\right|_{\substack{u=2\\ v=\frac{\pi}{4}}} = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = sen v \quad \Rightarrow \quad \left.\frac{\partial y}{\partial u}\right|_{\substack{u=2\\ v=\frac{\pi}{4}}} = \frac{\sqrt{2}}{2}$$

$$\frac{\partial z}{\partial u} = \left[4\sqrt{2}\,ln\left(\sqrt{2}\right)\right]\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)$$

$$\frac{\partial z}{\partial u} = 2\sqrt{2}\,ln\left(\sqrt{2}\right) + 2\sqrt{2} = \sqrt{2}\,ln\left(2\right) + 2\sqrt{2}$$

$$\boxed{\frac{\partial z}{\partial u} = \sqrt{2}\left(ln\left(2\right) + 2\right)}$$

16 Puntos

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10. Sea

$$V\left(x,y,z\right) = 5x^2 - 3xy + xyz \qquad\qquad P\left(3,4,5\right)$$

$$D_{\hat{a}}\,V\left(x,y,z\right) = \nabla V \cdot \hat{a}$$

$$\hat{a} = i + j - k \qquad\qquad \|\hat{a}\| = \sqrt{1^2 + 1^2 + \left(-1\right)^2}$$

$$\hat{a} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$\nabla V = \frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k$$

$$\left.\frac{\partial V}{\partial x}\right|_P = 10x - 3y + yz = 10\left(3\right) - 3\left(4\right) + 4\left(5\right)$$

$$= 30 - 12 + 20 = 38$$

$$\left.\frac{\partial V}{\partial y}\right|_P = -3x + xz = -3\left(3\right) + 3\left(5\right) = -9 + 15 = 6$$

$$\left.\frac{\partial V}{\partial z}\right|_P = xy = 3(4) = 12$$

$$\nabla V = 38i + 6j + 12k$$

$$D_{\hat{a}} V(x,y,z) = (38, 6, 12)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$D_{\hat{a}} V(x,y,z) = \frac{1}{\sqrt{3}}(38 + 6 - 12) = \frac{32}{\sqrt{3}}$$

$$\boxed{D_{\hat{a}} V(x,y,z) = \frac{32}{\sqrt{3}}}$$

$b)$ $\boxed{\nabla V = 38i + 6j + 12k}$

20 Puntos